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Exact asymptotic behaviour of fermion correlation functions in the massive Thirring model

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Abstract

We obtain an exact asymptotic expression for the two-point fermion correlation functions in the massive Thirring model (MTM) and show that, for $\beta^2 = 8\pi$, they reproduce the exactly known corresponding functions of the massless theory, explicitly confirming the irrelevance of the mass term at this point. This result is obtained by using the Coulomb gas representation of the fermionic MTM correlators in the bipolar coordinate system.

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1. Introduction

In a recent paper [1], we have presented the first exact evaluation of the Kosterlitz–Thouless (KT) critical exponent appearing in the asymptotic large distance behaviour of the two-point spin correlation function of the XY-model (which is the name given for the system consisting of planar spins interacting through an exchange coupling in a lattice). This has been done by using its connection to the sine-Gordon (SG) theory [2, 3] and the two-dimensional (2D) neutral Coulomb gas (CG) [4] expressed in bipolar coordinates, which allow us to obtain a convenient representation for the relevant correlator.

In this work we employ the same methodology established in [1] to compute the two-point fermion correlation functions of the MTM, which, as is well known, is also connected to the SG theory [5]. The MTM and the associated SG theory, indeed, are some of the best-studied quantum field theories. Numerous nontrivial exact results have been obtained for this fascinating system. Among them, we may list: the demonstration of the identity between the vacuum functionals of the MTM and SG theory [5]; the identification of the fermionic MTM field as the soliton operator of the SG theory [5, 6]; the explicit obtainment of an expression for this field operator in terms of the SG field (bosonization) [6]; the exact S-matrix and spectrum of bound-states [7]; recent investigations on new aspects of the relationship between the 2D Thirring model and the SG theory [8–10]; the determination of the free energy and the specific heat of the system by means of the thermodynamic Bethe ansatz [11]; and, finally, the derivation of exact form factors for the soliton operators and other fields [12–16] and its consequent use for the computation of density correlation functions [17]. Several exact results concerning the equilibrium statistical mechanics of the 2D classical CG have also been obtained [18]. Among these, we mention the full thermodynamics for $0 \leqslant \beta^2 < 4\pi$, in the case of point particles [19] and for $4\pi \leqslant \beta^2 < 6\pi$, in the case of extensive ones [20]. Charge and particle correlators have been obtained in the low temperature ($\beta^2 > 8\pi$) phase [21].

Renormalization group analysis of the SG/CG system has also produced numerous interesting results. It has been shown, in particular, that the mass term of the MTM or, equivalently, the cosine interaction of the SG theory, becomes irrelevant for $\beta^2 \geqslant 8\pi$ [22, 23]. The continuous phase transition of Kosterlitz and Thouless [24] was identified in the associated XY-model of spins at the temperature corresponding to this value of the SG coupling β and the associated critical exponent was evaluated by using scaling arguments and the irrelevance of the corresponding interaction [2, 22, 25, 26].

Despite this huge mass of important results, however, the fermion field correlation functions of the MTM are not known exactly, except for the special point $\beta^2 = 4\pi$ [27], where the MTM becomes a free massive theory. It is the purpose of this work to obtain an exact large distance asymptotic expression for the fermion correlators of the MTM and to show that for $\beta^2 = 8\pi$ (KT critical point) this asymptotic behaviour reproduces the exactly known fermion correlation functions of the massless Thirring model [28], thus confirming explicitly the irrelevance of the mass term of the MTM at this point. In order to do that, we make use of the CG representation of the SG system and the bosonized form of the MTM fields, which in the CG framework become associated with external charges and strings of electric dipoles interacting with the charges of the gas. The exact large distance asymptotic form of the correlators is then obtained by the use of a special coordinate system, namely the bipolar coordinates.

2. The MTM and the SG/CG system

In this section, we are going to review some basic features of the connection of the MTM with the SG theory and with the 2D neutral CG. We then finish by presenting a representation of the fermionic MTM correlators in the framework of the classical CG, which will be our starting point for their evaluation at $\beta^2 = 8\pi$.

The MTM is described by the Lagrangian

$$\mathcal{L} = \mathrm{i}\bar{\psi}\ \partial\!\!\!/\psi - M_0\bar{\psi}\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi), \tag{1}$$

where ψ is a two-component Dirac fermion field in (1+1)D. It is well known that it can be mapped into the SG theory of an scalar field [5] whose dynamics is determined by

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + 2\alpha_0\cos\beta\phi, \tag{2}$$

where the couplings in the two models are related as

$$g = \pi\left(\frac{4\pi}{\beta^2} - 1\right), \qquad M_0\bar{\psi}\psi = -2\alpha_0\cos\beta\phi. \tag{3}$$

Under this mapping, the two components of the fermion field may be expressed in terms of the SG field as

$$\psi_1(\vec{x}) = \sigma(\vec{x})\mu(\vec{x}), \qquad \psi_2(\vec{x}) = \sigma^\dagger(\vec{x})\mu(\vec{x}), \tag{4}$$

where $\sigma(\vec{x})$ and $\mu(\vec{x})$ are, respectively, order and disorder fields, satisfying a dual algebra, which can be introduced in the SG theory [30]. These are given by

$$\sigma(x,\tau) = \exp\left\{\mathrm{i}\,\frac{\beta}{2}\phi(x,\tau)\right\}, \tag{5}$$

$$\mu(x,\tau) = \exp\left\{\mathrm{i}\,\frac{2\pi}{\beta}\int_{-\infty}^{x}\mathrm{d}z\,\dot{\phi}(z,\tau)\right\}. \tag{6}$$

Equation (4) coincides with the bosonized expression for the fermion field, first obtained in [6].

In what follows, we are going to perform an expansion in α_0. In order to control the infrared (IR) divergences inherent to the expansion around a massless theory in 2D, we follow [5] and modify (2) by adding a regulator mass term

$$\mathcal{L}_{\mathrm{reg}} = \tfrac{1}{2}\mu_0^2\phi^2 \tag{7}$$

and multiplying the interaction term $(2\alpha_0\cos\beta\phi)$ by a function $f(\vec{z})$ of compact support. At the end, of course, we must take the limits $\mu_0 \to 0$ and $f(\vec{z}) \to 1$.

The Euclidean vacuum functional of the SG theory may be written as the grand-partition function of a classical neutral 2D CG, namely [4, 22]

$$\mathcal{Z} = \lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(n!)^2}\int\prod_{i=1}^{2n}(\mathrm{d}^2 z_i\, f(\vec{z}_i))$$
$$\times\exp\left\{\frac{\beta^2}{8\pi}\sum_{i\neq j=1}^{2n}\lambda_i\lambda_j\ln\left[\mu_0^2(|\vec{z}_i-\vec{z}_j|^2+|\varepsilon|^2)\right]\right\}, \tag{8}$$

where $\lambda_i = 1$ for $1 \leqslant i \leqslant n$ and $\lambda_i = -1$ for $n+1 \leqslant i \leqslant 2n$ and ε is an ultraviolet (UV) regulator, introduced in the 2D Coulomb potential, which is needed in the case of point particles or, equivalently, of a local field theory. The renormalized coupling α is related to the one in (2) by

$$\alpha = \alpha_0\left(\mu_0^2|\varepsilon|^2\right)^{\frac{\beta^2}{8\pi}}. \tag{9}$$

We must emphasize that, in order to obtain (8), use was made of the UV- and IR-regulated Green's function of the free scalar theory, namely [22]

$$G(\vec{r};\mu_0) = \frac{1}{2\pi}\mathrm{K}_0\left[\mu_0(|\vec{r}|^2+|\varepsilon|^2)^{\frac{1}{2}}\right] \overset{\mu_0|\vec{r}|\ll 1}{\sim} -\frac{1}{4\pi}\ln\left[\mu_0^2(|\vec{r}|^2+|\varepsilon|^2)\right], \tag{10}$$

where K_0 is a Bessel function.

Note that, due to neutrality, the explicit dependence on μ_0 disappears from the summand in (8).

In the CG language, the couplings α and β are related, respectively, to the CG fugacity and temperature as $\alpha = \mu_{\mathrm{CG}}$ and $\beta^2 = \frac{2\pi}{k_B T_{\mathrm{CG}}}$. At the Kosterlitz–Thouless point T_{KT}, corresponding to $\beta^2 = 8\pi$, the system undergoes a phase transition from a metallic (fluid) phase composed of charged particles, for $\beta^2 < 8\pi$, to an insulating (dielectric) phase, composed of neutral dipoles, for $\beta^2 > 8\pi$. In the region $0 < \beta^2 < 4\pi$, the singularities occurring in (8) are all integrable and no UV regularization is needed for Green's function. The CG of point charges is thermodynamically stable. For $4\pi \leqslant \beta^2 < 8\pi$, however, the singularities are no longer integrable and the system becomes unstable. Use of the UV-regularized Green's function introduced above becomes therefore necessary, in order to prevent thermodynamic collapse.

Using the CG description we can write the four components of the two-point fermion correlation function as

$$
\begin{aligned}
\left\langle\psi_{1(2)}(\vec{x})\psi^{\dagger}_{1(2)}(\vec{y})\right\rangle = \lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0}\frac{+(-)\mathrm{i}\exp\left[+(-)\mathrm{i}\arg\left(\vec{x}-\vec{y}\right)\right]}{\mathcal{Z}}\left[\frac{|\varepsilon|}{|\vec{x}-\vec{y}|}\right]^{\left(\frac{2\pi}{\beta^2}+\frac{\beta^2}{8\pi}\right)} \\
\times\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(n!)^2}\int\prod_{i=1}^{2n}(\mathrm{d}^2 z_i\, f(\vec{z}_i))\exp\left\{\frac{\beta^2}{8\pi}\sum_{i\neq j=1}^{2n}\lambda_i\lambda_j\ln\left[\mu_0^2(|\vec{z}_i-\vec{z}_j|^2+|\varepsilon|^2)\right]\right. \\
\left.+(-)\frac{\beta^2}{8\pi}\sum_{i=1}^{2n}\lambda_i\ln\frac{[|\vec{z}_i-x|^2+|\varepsilon|^2]}{[|\vec{z}_i-y|^2+|\varepsilon|^2]}+\mathrm{i}\sum_{i=1}^{2n}\lambda_i[\arg(\vec{z}_i-\vec{y})-\arg(\vec{z}_i-\vec{x})]\right\}
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
\left\langle\psi_{1(2)}(\vec{x})\psi^{\dagger}_{2(1)}(\vec{y})\right\rangle = \lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0}\frac{-(+)\mathrm{i}}{\mathcal{Z}}[\mu_0|\varepsilon|]^{\left(\frac{2\pi}{\beta^2}+\frac{\beta^2}{8\pi}\right)}[\mu_0|\vec{x}-\vec{y}|]^{-\left(\frac{2\pi}{\beta^2}-\frac{\beta^2}{8\pi}\right)} \\
\times\sum_{n=0}^{\infty}\frac{\alpha^{(2n+1)}}{n!(n+1)!}\int\prod_{i=1}^{2n+1}(\mathrm{d}^2 z_i\, f(\vec{z}_i)) \\
\times\exp\left\{\frac{\beta^2}{8\pi}\sum_{i\neq j=1}^{2n+1}\lambda_i\lambda_j\ln\left[\mu_0^2(|\vec{z}_i-\vec{z}_j|^2+|\varepsilon|^2)\right]\right. \\
+(-)\frac{\beta^2}{8\pi}\sum_{i=1}^{2n+1}\lambda_i\ln\left\{\left[\mu_0^2(|\vec{z}_i-\vec{x}|^2+|\varepsilon|^2)\right]\left[\mu_0^2(|\vec{z}_i-\vec{y}|^2+|\varepsilon|^2)\right]\right\} \\
\left.+\,\mathrm{i}\sum_{i=1}^{2n+1}\lambda_i[\arg(\vec{z}_i-\vec{y})-\arg(\vec{z}_i-\vec{x})]\right\}.
\end{aligned}
\tag{12}
$$

In (11), we have $\lambda_i = +1$ for $1 \leqslant i \leqslant n$ and $\lambda_i = -1$ for $n+1 \leqslant i \leqslant 2n$. In (12), on the other hand, we have $\lambda_i = +1$ for $1 \leqslant i \leqslant n$ and $\lambda_i = -1$ for $n+1 \leqslant i \leqslant 2n+1$ for $\langle\psi_1\psi_2^{\dagger}\rangle$. Conversely, for $\langle\psi_2\psi_1^{\dagger}\rangle$, we have $\lambda_i = -1$ for $1 \leqslant i \leqslant n$ and $\lambda_i = +1$ for $n+1 \leqslant i \leqslant 2n+1$.

Note that the order fields introduce additional external charges of half magnitude in the gas. The disorder fields, on the other hand, introduce strings of electric dipoles connecting $\vec{x}$ and $\vec{y}$ and whose interaction potential with a charge at $\vec{z}$ is proportional to $(\arg(\vec{z}-\vec{y})-\arg(\vec{z}-\vec{x}))$ [30]. As a consequence, in the case of the diagonal components of the fermionic correlators, we have two external charges with half of the magnitude of the gas charges and *opposite* signs, located at $\vec{x}$ and $\vec{y}$. The CG, therefore, remains neutral. In the case of the off-diagonal components, however, the fermion fields introduce two external charges, also having half magnitude and with the *same* sign at $\vec{x}$ and $\vec{y}$. In order to achieve global neutrality, therefore, the CG must be no longer neutral, having an extra positive charge in the case of $\langle\psi_2\psi_1^{\dagger}\rangle$ and an extra negative charge in the case of $\langle\psi_1\psi_2^{\dagger}\rangle$. Global neutrality is a necessary condition for the existence of the $\mu_0 \to 0$ limit, since in this case the μ_0-factors are completely cancelled in (11) and (12).

3. Bipolar coordinates and the fermion correlators

In this section we make use of bipolar coordinates in order to obtain a representation for the fermion correlators that will prove to be extremely useful. It allows, in particular, the obtainment of a series in $|\vec{x}-\vec{y}|$ for these correlators, valid for $\vec{x}\neq\vec{y}$, out of which we can derive an exact asymptotic expression at $\beta^2 = 8\pi$.

Given the position vector $\vec{r}$ in the plane and two points (poles) at $\vec{x}$ and $\vec{y}$, the bipolar coordinates (ξ,η) are defined as [31]

$$\xi = \arg(\vec{r}-\vec{y}) - \arg(\vec{r}-\vec{x}), \qquad \eta = \ln\frac{|\vec{r}-\vec{x}|}{|\vec{r}-\vec{y}|}, \tag{13}$$

with $0\leqslant\xi\leqslant 2\pi$ and $-\infty<\eta<\infty$. In terms of these coordinates, the position vector is given by

$$\vec{r} = \frac{|\vec{x}-\vec{y}|}{2[\cosh\eta-\cos\xi]}(\sinh\eta, \sin\xi) \tag{14}$$

and the volume element reads

$$\mathrm{d}^2 z = \frac{|\vec{x}-\vec{y}|^2}{4[\cosh\eta-\cos\xi]^2}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{15}$$

Rewriting expressions (11) and (12), for $\vec{x}\neq\vec{y}$, in terms of bipolar coordinates, we get

$$\begin{aligned}
\langle\psi_{1(2)}(\vec{x})\psi^{\dagger}_{1(2)}(\vec{y})\rangle &= \lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0}\frac{+(-)\mathrm{i}\exp[+(-)\mathrm{i}\arg(\vec{x}-\vec{y})]}{\mathcal{Z}}\left[\frac{|\varepsilon|}{|\vec{x}-\vec{y}|}\right]^{(\frac{2\pi}{\beta^2}+\frac{\beta^2}{8\pi})} \\
&\times\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(n!)^2}\int_{0,V(\varepsilon)}^{2\pi}\int_{-\infty,V(\varepsilon)}^{+\infty}\prod_{i=1}^{2n}(\mathrm{d}\xi_i\,\mathrm{d}\eta_i\,f(\xi_i,\eta_i)) \\
&\times\frac{|\vec{x}-\vec{y}|^{4n}}{4[\cosh\eta_i-\cos\xi_i]^2}\exp\left\{\frac{\beta^2}{8\pi}\sum_{i\neq j=1}^{2n}\lambda_i\lambda_j\right. \\
&\times\ln\left\{[\mu_0|\vec{x}-\vec{y}|]^2\left[\left(\frac{\sinh\eta_i}{2[\cosh\eta_i-\cos\xi_i]}-\frac{\sinh\eta_j}{2[\cosh\eta_j-\cos\xi_j]}\right)^2\right.\right. \\
&\left.\left.+\left(\frac{\sin\xi_i}{2[\cosh\eta_i-\cos\xi_i]}-\frac{\sin\xi_j}{2[\cosh\eta_j-\cos\xi_j]}\right)^2\right]\right\} \\
&\left.+(-)\frac{\beta^2}{4\pi}\sum_{i=1}^{2n}\lambda_i\eta_i+\mathrm{i}\sum_{i=1}^{2n}\lambda_i\xi_i\right\},
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\langle\psi_{1(2)}(\vec{x})\psi^{\dagger}_{2(1)}(\vec{y})\rangle &= \lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0}\frac{-(+)\mathrm{i}}{\mathcal{Z}}[\mu_0|\varepsilon|]^{(\frac{2\pi}{\beta^2}+\frac{\beta^2}{8\pi})}[\mu_0|\vec{x}-\vec{y}|]^{-(\frac{2\pi}{\beta^2}-\frac{\beta^2}{8\pi})} \\
&\times\sum_{n=0}^{\infty}\frac{\alpha^{(2n+1)}}{n!(n+1)!}\int_{0,V(\varepsilon)}^{2\pi}\int_{-\infty,V(\varepsilon)}^{+\infty}\prod_{i=1}^{2n+1}(\mathrm{d}\xi_i\,\mathrm{d}\eta_i\,f(\xi_i,\eta_i)) \\
&\times\frac{|\vec{x}-\vec{y}|^{(4n+2)}}{4[\cosh\eta_i-\cos\xi_i]^2}\exp\left\{\frac{\beta^2}{8\pi}\sum_{i\neq j=1}^{2n+1}\lambda_i\lambda_j\right. \\
&\times\ln\left\{[\mu_0|\vec{x}-\vec{y}|]^2\left[\left(\frac{\sinh\eta_i}{2[\cosh\eta_i-\cos\xi_i]}-\frac{\sinh\eta_j}{2[\cosh\eta_j-\cos\xi_j]}\right)^2\right.\right. \\
&\left.\left.+\left(\frac{\sin\xi_i}{2[\cosh\eta_i-\cos\xi_i]}-\frac{\sin\xi_j}{2[\cosh\eta_j-\cos\xi_j]}\right)^2\right]\right\}
\end{aligned}$$

$$+(-)\frac{\beta^2}{8\pi}\sum_{i=1}^{2n+1}\lambda_i\ln\left\{[\mu_0|\vec{x}-\vec{y}|]^4\left[\left(\frac{\sinh\eta_i}{2[\cosh\eta_i-\cos\xi_i]}+\frac{1}{2}\right)^2\right.\right.$$
$$\left.+\left(\frac{\sin\xi_i}{2[\cosh\eta_i-\cos\xi_i]}\right)^2\right]\left[\left(\frac{\sinh\eta_i}{2[\cosh\eta_i-\cos\xi_i]}-\frac{1}{2}\right)^2$$
$$\left.\left.+\left(\frac{\sin\xi_i}{2[\cosh\eta_i-\cos\xi_i]}\right)^2\right]\right\}+\mathrm{i}\sum_{i=1}^{2n+1}\lambda_i\xi_i\Bigg\}. \tag{17}$$

Note that in the two previous expressions, we modified the UV regulating method, by redefining the integration region as $V(\varepsilon)$, in such a way that the integrations must respect the condition $|\vec{z}_i-\vec{z}_j|>\varepsilon$. In terms of the ξ_i, η_i integrals, this implies the following restriction for the expressions between round brackets in (16) and (17), which we call, respectively, α_{ij} and β_{ij}:

$$\left[\alpha_{ij}^2+\beta_{ij}^2\right]>\frac{|\varepsilon|^2}{|\vec{x}-\vec{y}|^2}. \tag{18}$$

It is easy to see that the $|\vec{x}-\vec{y}|$-factors decouple from the integrals in (16) and (17). We can also see that, for finite $|\vec{x}-\vec{y}|$, the μ_0-factors completely cancel out from the fermion correlators, as we observed at the end of the previous section.

A simple combinatoric analysis, considering the neutrality of the system, shows that for the diagonal components of the correlation functions, the $|\vec{x}-\vec{y}|^{\frac{\beta^2}{2\pi}}$-factors appear $n(n-1)$ times in the numerator and n^2 times in the denominator. Adding the $4n$ contribution coming from the scale factors of the volume elements, we obtain

$$\left\langle\psi_{1(2)}(\vec{x})\psi_{1(2)}^{\dagger}(\vec{y})\right\rangle=\lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0}\frac{+(-)\mathrm{i}\exp\left[+(-)\mathrm{i}\arg\left(\vec{x}-\vec{y}\right)\right]}{\mathcal{Z}}\left[\frac{|\varepsilon|}{|\vec{x}-\vec{y}|}\right]^{\left(\frac{2\pi}{\beta^2}+\frac{\beta^2}{8\pi}\right)}$$
$$\times\sum_{n=0}^{\infty}C_n^{+(-)}\left(|\vec{x}-\vec{y}|\right)|\vec{x}-\vec{y}|^{\left(2-\frac{\beta^2}{4\pi}\right)2n}, \tag{19}$$

where the $C_n^{+(-)}$ coefficients are given by each term of the summand in (16) after the removal of the $|\vec{x}-\vec{y}|$-factors.

Conversely, for the off-diagonal components, a similar combinatoric analysis indicates that the $|\vec{x}-\vec{y}|^{\frac{\beta^2}{2\pi}}$-factors appear $n(n+1)$ times in the numerator and $(n+1)^2$ times in the denominator. Considering the $(4n+2)$ scale factors of the volume elements in (17), we get

$$\left\langle\psi_{1(2)}(\vec{x})\psi_{2(1)}^{\dagger}(\vec{y})\right\rangle=\lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0}\frac{-(+)\mathrm{i}}{\mathcal{Z}}\left[\mu_0|\varepsilon|\right]^{\left(\frac{2\pi}{\beta^2}+\frac{\beta^2}{8\pi}\right)}\left[\mu_0|\vec{x}-\vec{y}|\right]^{-\left(\frac{2\pi}{\beta^2}-\frac{\beta^2}{8\pi}\right)}$$
$$\times\sum_{n=0}^{\infty}F_n^{+(-)}\left(|\vec{x}-\vec{y}|\right)|\vec{x}-\vec{y}|^{\left[\left(2-\frac{\beta^2}{4\pi}\right)2n-\frac{\beta^2}{2\pi}+2\right]}, \tag{20}$$

where the $F_n^{+(-)}$ coefficients are given by each term of the summand in (17) after the removal of the $|\vec{x}-\vec{y}|$-factors.

Note that the coefficients $C_n^{+(-)}$ and $F_n^{+(-)}$ in (19) and (20) depend on $|\vec{x}-\vec{y}|$ through the restriction on the integration region given by (18).

In the limit $\alpha\to 0$, our expressions for the fermion correlation functions of the MTM reproduce the exact solution for the Euclidean correlators of the massless Thirring model. In the limit $\beta \to 0$, the mass operator becomes trivial and, again, we must take $\alpha \to 0$, thereby recovering the exact correlation functions of the massless Thirring model.

It is also interesting to note that our expressions for the two-point fermion correlators of the MTM, (19) and (20), reproduce, in the case $\beta^2 = 4\pi$ (free-fermion point) the free massive fermion correlation function

$$\langle \psi(\vec{x})\psi^\dagger(\vec{y})\rangle_0 = M_0 \begin{pmatrix} \zeta \mathrm{K}_1(M_0|\vec{x}-\vec{y}|) & \mathrm{K}_0(M_0|\vec{x}-\vec{y}|) \\ \mathrm{K}_0(M_0|\vec{x}-\vec{y}|) & \zeta^* \mathrm{K}_1(M_0|\vec{x}-\vec{y}|) \end{pmatrix}, \tag{21}$$

where $\zeta = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\arg(\vec{x}-\vec{y})}$ and M_0 is the free fermion mass. Indeed, for $\beta^2 = 4\pi$ the series appearing in (19) and (20), respectively, are precisely the ones that occur in the definition of the Bessel functions K_1 and K_0 [29]. Equating the coefficients we obtain the following expressions for $C_n^{+(-)}$ and $F_n^{+(-)}$ ($C_0^{+(-)} \equiv 1$ for all values of β):

$$C_{n+1}^{+(-)}(|\vec{x}-\vec{y}|) = \frac{{M_0}^{(2n+2)}}{2^{(2n+1)} n!(n+1)!}\left[\ln\left(\frac{M_0|\vec{x}-\vec{y}|}{2}\right) - \frac{1}{2}\psi(n+1) - \frac{1}{2}\psi(n+2)\right] \tag{22}$$

and

$$F_n^{+(-)}(|\vec{x}-\vec{y}|) = \frac{+(-)\mathrm{i}{M_0}^{2n}}{2^{2n}(n!)^2}\left[\psi(n+1) - \ln\left(\frac{M_0|\vec{x}-\vec{y}|}{2}\right)\right], \tag{23}$$

where $\psi(x)$ is the Euler function. In order to obtain (22) and (23), we used the fact that (19) and (20) are independent of μ_0. We then replaced μ_0 for the physical mass M_0 and eliminated the $|\varepsilon|$- and $\mathcal{Z}$-factors by renormalizing the fermion fields.

From the exact expression (19), we clearly see a definite change in the large distance behaviour of the diagonal correlation functions at $\beta^2 = 8\pi$. This indicates that for $\beta^2 > 8\pi$, the asymptotic large distance behaviour is determined by the corresponding massless correlator. We are going to see, in the next section, that for these values of the coupling constant β, also the off-diagonal components of the correlator at large distance, correspond to the respective massless correlators, namely, they vanish identically.

4. Asymptotic behaviour of fermion correlators

4.1. Diagonal components

Let us study in this subsection the large distance limit of the diagonal, chirality conserving, components of the fermion correlation function at $\beta^2 = 8\pi$. From (19), we can write

$$\left\langle \psi_{1(2)}(\vec{x})\psi^\dagger_{1(2)}(\vec{y})\right\rangle = \lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0} +(-)\mathrm{i}\exp[+(-)\mathrm{i}\arg(\vec{x}-\vec{y})] \times \left[\frac{|\varepsilon|}{|\vec{x}-\vec{y}|}\right]^{\frac{5}{4}} K^{+(-)}(|\vec{x}-\vec{y}|), \tag{24}$$

where

$$K^{+(-)}(|\vec{x}-\vec{y}|) = \mathcal{Z}^{-1}\sum_{n=0}^{\infty} C_n^{+(-)}(|\vec{x}-\vec{y}|) \tag{25}$$

evaluated at $\beta^2 = 8\pi$.

Going back to the original coordinate system and defining the symbols

$$[x_i, y_j] \equiv \mu_0^4[|\vec{x}_i - \vec{y}_j|^2 + |\varepsilon|^2]^2, \tag{26}$$

we may express $K^{+(-)}(|\vec{x}-\vec{y}|)$ in the form

$$K^{+(-)}(|\vec{x}-\vec{y}|) = \mathcal{Z}^{-1}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(n!)^2}\int\prod_{i=1}^{n}(\mathrm{d}^2x_i f(\vec{x}_i))\prod_{i=1}^{n}(\mathrm{d}^2y_i f(\vec{y}_i))\frac{\prod_{i<j}^{n}\frac{[x_i,x_j]}{|\vec{x}-\vec{y}|^4}\prod_{i<j}^{n}\frac{[y_i,y_j]}{|\vec{x}-\vec{y}|^4}}{|\vec{x}-\vec{y}|^{4n}\prod_{i,j}^{n}\frac{[x_i,y_j]}{|\vec{x}-\vec{y}|^4}}$$
$$\times\left(\frac{\prod_i^n[x_i,x]\prod_i^n[y_i,y]}{\prod_i^n[x_i,y]\prod_i^n[y_i,x]}\right)^{+(-)\frac{1}{2}}\exp\left\{\mathrm{i}\sum_{i=1}^{n}[\arg(\vec{x}_i-\vec{y})-\arg(\vec{x}_i-\vec{x})]\right.$$
$$\left.-\mathrm{i}\sum_{i=1}^{n}[\arg(\vec{y}_i-\vec{y})-\arg(\vec{y}_i-\vec{x})]\right\}, \tag{27}$$

where we went back to the original UV regulating method and associated the positive charges with $\vec{x}_i$ and the negative ones with $\vec{y}_i$. Observe that the $|\vec{x}-\vec{y}|$-factors completely cancel out in (27) and, therefore, may be removed.

Let us now study the asymptotic large distance behaviour of (24). In order to do that we must rewrite the expression between round brackets, as well as the phases, in (27) in the form that we would have obtained if we had used the fully regulated form of Green's function, given in (10), since in this limit the last part of that expression is no longer valid. Thus, we should write the expression between round brackets as

$$\exp\left\{-(+)2\left[\sum_{i=1}^{n}(\mathrm{K}_0[\mu_0(|\vec{x}_i-\vec{x}|^2+|\varepsilon|^2)^{\frac{1}{2}}]-\mathrm{K}_0[\mu_0(|\vec{x}_i-\vec{y}|^2+|\varepsilon|^2)^{\frac{1}{2}}])\right.\right.$$
$$\left.\left.-\sum_{i=1}^{n}(\mathrm{K}_0[\mu_0(|\vec{y}_i-\vec{x}|^2+|\varepsilon|^2)^{\frac{1}{2}}]-\mathrm{K}_0[\mu_0(|\vec{y}_i-\vec{y}|^2+|\varepsilon|^2)^{\frac{1}{2}}])\right]\right\}, \tag{28}$$

whereas, for the phases, we get

$$\int_{\vec{y}}^{2\vec{y}-\vec{x}}\mathrm{d}\xi_\mu\epsilon^{\mu\nu}\partial_\nu^{(\xi)}\left(\sum_{i=1}^{n}\mathrm{K}_0\big[\mu_0(|\vec{\xi}-\vec{x}_i+(\vec{x}-\vec{y})|^2+|\varepsilon|^2)^{\frac{1}{2}}\big]\right.$$
$$\left.-\sum_{i=1}^{n}\mathrm{K}_0\big[\mu_0(|\vec{\xi}-\vec{y}_i+(\vec{x}-\vec{y})|^2+|\varepsilon|^2)^{\frac{1}{2}}\big]\right), \tag{29}$$

where we have performed the shift $\vec{\xi}\to\vec{\xi}-(\vec{x}-\vec{y})$ in the integration variable. Note that the former expression for the phases may be obtained from (29), for $\mu_0|\vec{r}|\ll 1$, by using (10) and the Cauchy–Riemann equation for the logarithm function [30].

It is easy to see that, for $|\vec{x}-\vec{y}|\to\infty$, we have

$$[x_i,x]\overset{|\vec{x}-\vec{y}|\to\infty}{\sim}[x_i,y], \qquad [y_i,x]\overset{|\vec{x}-\vec{y}|\to\infty}{\sim}[y_i,y] \tag{30}$$

and therefore, the expression (28) tends to 1. Using the fact that $\mathrm{K}_0(x)\overset{x\to\infty}{\longrightarrow}0$, we may also see that the phases (29) vanish in the large distance limit. Consequently, considering that the remaining terms in the summand in (27) are identical to those in $\mathcal{Z}$, we get

$$K^{+(-)}(|\vec{x}-\vec{y}|)\overset{|\vec{x}-\vec{y}|\to\infty}{\longrightarrow}1. \tag{31}$$

The IR regulator, μ_0, as well as the functions $f(\vec{z})$, can now be safely removed in (24). Introducing the renormalized fields

$$\psi_{1(2)}^{\mathrm{R}}=\psi_{1(2)}|\varepsilon|^{-\frac{5}{8}}, \tag{32}$$

also the UV regulator ε may be removed in (24) and we, finally, obtain

$$\langle\psi_{1(2)}(\vec{x})\psi_{1(2)}^{\dagger}(\vec{y})\rangle_{\mathrm{R}}\overset{|\vec{x}-\vec{y}|\to\infty}{\sim}\frac{+(-)\mathrm{i}\exp[+(-)\mathrm{i}\arg(\vec{x}-\vec{y})]}{|\vec{x}-\vec{y}|^{\frac{5}{4}}}, \tag{33}$$

which are the diagonal components of the Euclidean correlator corresponding to Klaiber's exact solution of the massless Thirring model [28].

4.2. Off-diagonal components

We now consider the off-diagonal, chirality nonconserving, components of the fermion correlation function at $\beta^2 = 8\pi$. From (20), we may write

$$\left\langle \psi_{1(2)}(\vec{x})\psi^{\dagger}_{2(1)}(\vec{y})\right\rangle = \lim_{\varepsilon\to 0}\lim_{f(z)\to 1}\lim_{\mu_0\to 0} -(+)\mathrm{i}[\mu_0|\varepsilon|]^{\frac{5}{4}}[\mu_0|\vec{x}-\vec{y}|]^{\frac{3}{4}} G^{+(-)}(|\vec{x}-\vec{y}|)\,, \tag{34}$$

where

$$G^{+(-)}(|\vec{x}-\vec{y}|) = \mathcal{Z}^{-1}|\vec{x}-\vec{y}|^{-2}\sum_{n=0}^{\infty} F_n^{+(-)}(|\vec{x}-\vec{y}|) \tag{35}$$

evaluated at $\beta^2 = 8\pi$.

Proceeding as before, we go back to the original coordinate system and express G^+ as

$$\begin{aligned} G^+(|\vec{x}-\vec{y}|) = \mathcal{Z}^{-1}|\vec{x}-\vec{y}|^{-2}\sum_{n=0}^{\infty}\frac{\alpha^{(2n+1)}}{n!(n+1)!}\int\prod_{i=1}^{n}(\mathrm{d}^2x_i\, f(\vec{x}_i))\prod_{i=1}^{n+1}(\mathrm{d}^2y_i\, f(\vec{y}_i)) \\ \times\frac{\prod_{i<j}^{n}\frac{[x_i,x_j]}{|\vec{x}-\vec{y}|^4}\prod_{i<j}^{n+1}\frac{[y_i,y_j]}{|\vec{x}-\vec{y}|^4}}{|\vec{x}-\vec{y}|^{(4n+2)}\prod_i^n\prod_j^{n+1}\frac{[x_i,y_j]}{|\vec{x}-\vec{y}|^4}}\left(\frac{\prod_i^n\frac{[x_i,x]}{|\vec{x}-\vec{y}|^4}\prod_i^n\frac{[x_i,y]}{|\vec{x}-\vec{y}|^4}}{\prod_i^{n+1}\frac{[y_i,x]}{|\vec{x}-\vec{y}|^4}\prod_i^{n+1}\frac{[y_i,y]}{|\vec{x}-\vec{y}|^4}}\right)^{\frac{1}{2}} \\ \times\exp\left\{\mathrm{i}\sum_{i=1}^{n}[\arg(\vec{x}_i-\vec{y})-\arg(\vec{x}_i-\vec{x})]-\mathrm{i}\sum_{i=1}^{n+1}[\arg(\vec{y}_i-\vec{y})-\arg(\vec{y}_i-\vec{x})]\right\}. \end{aligned} \tag{36}$$

G^-, accordingly, may be obtained from (36) by just performing the exchange $x_{i(j)} \leftrightarrow y_{i(j)}$ and reversing the sign of the phases.

By inspecting (36), we immediately see that the $|\vec{x}-\vec{y}|$-factors completely cancel out and we conclude that $G^{+(-)}$ are dimensionless as they should. Counting the μ_0-factors in the above expression we also see that there is an overall μ_0^{-2}. Thus, inserting this result in (34), we conclude that the regulating mass (IR regulator) μ_0 completely disappears from the off-diagonal components of the fermion correlator. As we shall see below, however, this situation is modified when we consider the asymptotic behaviour of these functions.

We may now analyse the asymptotic large distance behaviour of the off-diagonal components of the fermion correlator, (34). As we saw in the case of (27), the phases in (36) will vanish in this limit. Taking this fact into account and shifting the integration variables as

$$\vec{x}_i \to \vec{x}_i - \vec{x}, \qquad \vec{y}_i \to \vec{y}_i - \vec{x}, \tag{37}$$

we can see from (36) that

$$\begin{aligned} G^+(|\vec{x}-\vec{y}|) \overset{|\vec{x}-\vec{y}|\to\infty}{\sim} -\mathcal{Z}^{-1}\sum_{n=0}^{\infty}\frac{\alpha^{(2n+1)}}{n!(n+1)!}\int\prod_{i=1}^{n}(\mathrm{d}^2x_i\ f(\vec{x}_i))\prod_{i=1}^{n+1}(\mathrm{d}^2y_i\ f(\vec{y}_i)) \\ \times\frac{\prod_{i<j}^{n}[x_i,x_j]\prod_{i<j}^{n+1}[y_i,y_j]}{\prod_i^n\prod_j^{n+1}[x_i,y_j]}\left(\frac{\prod_i^n[x_i,0]}{\prod_i^{n+1}[y_i,0]}\right)^{\frac{1}{2}}\left(\frac{\prod_i^n[x_i,y-x]}{\prod_i^{n+1}[y_i,y-x]}\right)^{\frac{1}{2}}. \end{aligned} \tag{38}$$

In the large distance regime, again, we must rewrite the last factor in the above expression in a form analogous to (28), in terms of the fully regulated Green's function, namely

$$\exp\left\{-(+)2\left[\sum_{i=1}^{n}\mathrm{K}_0\big[\mu_0(|\vec{x}_i-(\vec{y}-\vec{x})|^2+|\varepsilon|^2)^{\frac{1}{2}}\big]-\sum_{i=1}^{n+1}\mathrm{K}_0\big[\mu_0(|\vec{y}_i-(\vec{y}-\vec{x})|^2+|\varepsilon|^2)^{\frac{1}{2}}\big]\right]\right\}. \tag{39}$$

Using, as before, the fact that $\mathrm{K}_0(x)\stackrel{x\to\infty}{\longrightarrow}0$, we conclude that the above expression tends to 1 for $|\vec{y}-\vec{x}|\to\infty$. Counting the μ_0-factors in the remaining terms of (38), taking into account (9) and (26), we find that they completely cancel out. Therefore, inserting this result in (34) and renormalizing the fields as in (32), we get

$$\langle\psi_{1(2)}(\vec{x})\psi^{\dagger}_{2(1)}(\vec{y})\rangle_{\mathrm{R}}\stackrel{|\vec{x}-\vec{y}|\to\infty}{\sim}\lim_{\varepsilon\to0}\lim_{f(z)\to1}\lim_{\mu_0\to0}-(+)\mathrm{i}\mu_0^2|\vec{x}-\vec{y}|^{\frac{3}{4}}\kappa(\varepsilon)^{+(-)}, \tag{40}$$

where $\kappa(\varepsilon)^{+}$ is given by (38), after removing the last factor (note that this is independent of μ_0).

Using Coleman's prescription that the mass (IR) regulator should be eliminated first [5], we finally get

$$\langle\psi_{1(2)}(\vec{x})\psi^{\dagger}_{2(1)}(\vec{y})\rangle_{\mathrm{R}}\stackrel{|\vec{x}-\vec{y}|\to\infty}{\longrightarrow}0, \tag{41}$$

which coincides with the result for the off-diagonal components of the fermion correlator in the massless Thirring model. These components of the correlator vanish, in that case, because of chirality conservation that exists in a massless fermionic theory.

We can understand physically, in terms of the CG picture, the reason why the off-diagonal correlators vanish in the large distance regime. The neutrality of the gas is responsible for the complete cancellation of the IR regulator μ_0. In the case of the diagonal components, two external charges of *opposite* sign are introduced for the description of the correlation function. In the large distance regime, the external charges are removed to infinity and decouple from the gas, since the fully regulated 2D Coulomb interaction vanishes at large distances. Removing these charges to infinity leaves a gas that remains neutral. The μ_0-factors are totally cancelled out and the correlators are finite, as we can see from (33). Conversely, for the off-diagonal components, two external charges of *the same* sign are introduced in the system. These, together with the gas charges, form a neutral system. When we remove the external charges to infinity, for describing the large distance regime of the correlator, a non-neutral gas is left after the decoupling of these charges. Then, the IR regulator μ_0 no longer cancels out and forces the correlation functions to vanish for $\mu_0\to0$.

These results clearly expose the fact that the mass term of the MTM becomes irrelevant at $\beta^2=8\pi$.

5. Concluding remarks

We would like to comment on the prescription adopted concerning the regulators. When studying the asymptotic behaviour of the correlation functions, we always take the limit $|\vec{x}-\vec{y}|\to\infty$ firstly. Then, following [5], we take the regulators out in the order: (1) $\mu_0\to0$; (2) $f(\vec{z})\to1$ and (3) $\varepsilon\to0$. This leads, as we have seen, to the correct asymptotic limit of the massive fermion correlators. For finite $|\vec{x}-\vec{y}|$, conversely, we have shown that the μ_0 regulator completely cancels out and the limit $\mu_0\to0$ can be taken safely. Nevertheless, when removing the UV regulator ε, we must be careful because of the singularities that will appear due to the short-distance Coulomb interaction of point charges. This has been studied in detail for $4\pi\leqslant\beta^2<8\pi$ and it was shown that the singularities that appear at multipole thresholds may be absorbed by a subtractive renormalization of the ground-state energy [32]. Nevertheless, the conjectured existence of a sequence of phase transitions coinciding with these multipole thresholds in the region $4\pi \leqslant \beta^2 < 8\pi$ [33, 34], has been later on denied [35, 20]. For $\beta^2 \geqslant 8\pi$, however, the UV problem becomes extremely complicated and, as far as we know, remains unsolved. Consequently, only the large distance regime ($|\vec{x} - \vec{y}| \to \infty$) of the MTM, which has been studied here, can be considered sensible in this region of the coupling β.

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